Number Theory

1. Solution: Since $20^{18} = (2^2 5)^{18} = 2^{36} 5^{18}$, $a + b + x + y = 2 + 5 + 36 + 18 = 61$.

   Answer: (B)

2. Solution: Units digits repeat 3, 9, 7, 1, 3, 9, … So the units digit of 3^{2018} is 9.

   Answer: (D)

3. Solution: Note that $AB_7 = 7A + B = BA_5 = 5B + A$. From this we have $3A = 2B = 6$. So $A = 2, B = 3$.

   Answer: (A)

4. Solution: $M = 1936 = 44^2$ and $N = 2025 = 45^2$. So $N - M = 89$.

   Answer: (D)

5. Solution: A number represented in base 8 is divisible by 7 if and only if the digit sum is divisible by 7.

   Since $2 + 3 + 4 + 5 = 14$ is divisible by 7, 2345_8 is divisible by 7.

   Answer: (B)

6. Solution: $3^{12} - 1 = (3^6 - 1)(3^6 + 1) = (3^3 - 1)(3^3 + 1)(3^6 + 1) = (3 - 1)(3^2 + 3 + 1)(3 + 1)(3^2 - 3 + 1)(3^2 + 1)(3^4 - 3^2 + 1) = 2 \cdot 13 \cdot 4 \cdot 7 \cdot 10 \cdot 73$

   Answer: (C)

7. Solution: Note that $(x + 5)(y + 3) = 60$. There are four pairs of (x, y) from $\{x + 5 = 6, y + 3 = 10\}, \{x + 5 = 10, y + 3 = 6\}, \{x + 5 = 12, y + 3 = 5\}, \{x + 5 = 15, y + 3 = 4\}$.

   Answer: (B)

8. Solution: There are five such two-digit numbers: 72 and 96 (with two prime factors); 60, 84, 90 (with three prime factors)

   Answer: (D)

9. Solution: Note that $2 \cdot 4 \cdot 6 \cdot \cdots \cdot 98 \cdot 100 = 2^{50}(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots 50)$. Therefore, 47 is the largest prime factor

   Answer: (C)

10. Solution: Such numbers form an arithmetic sequence $a_1 = 108, a_2 = 143, \cdots, a_{26} = 983, a_{27} = 1018$

    Answer: (A)

11. Solution: There is only one pair of two prime numbers whose sum is 45. It is $(2,43)$. Then $m = 2 \cdot 43 = 86$.

    Answer: (D)

12. Solution: The Euler-Totient function $\phi(504) = \phi(7 \cdot 8 \cdot 9) = \phi(7)\phi(8)\phi(9) = 6 \cdot 4 \cdot 6 = 144$.

    Answer: (B)

13. Solution: Note that $24024 = 24 \cdot 1001 = 24 \cdot 7 \cdot 11 \cdot 13 = 11 \cdot 12 \cdot 13 \cdot 14$. So the sum of four integers is $11 + 12 + 13 + 14 = 50$.

    Answer: (A)

14. Solution: If $x > y$, then $x^2 - y^2 = (x-y)(x+y) = 72$. Since both $x - y$ and $x + y$ must be even numbers, there are 3 pairs of equations $\{x - y = 2, x + y = 36\}, \{x - y = 4, x + y = 18\}, \{x - y = 6, x + y = 12\}$ yielding solutions to the equation. There are three more pairs from the case $y > x$.

    Answer: (B)

15. Solution: The four-digit number $abba$ is a multiple of 9 if and only if the two digit number ab is a multiple of 9. So there are ten 4-digit palindromes that are multiples of 9. $abba$ is a multiple of 7 if and only if b is a multiple of 7 ($b = 0, or\ 7$). So there are 18 of multiples of 7. $10 + 18 = 28$

    Answer: (A)

16. Solution: Since $(b-1)^2$ and $(a+4)^2$ are perfect squares, $a^2 + 7$ must be a perfect square as well. In other words, both of two numbers, a^2 and $a^2 + 7$, 7 apart are perfect squares. This shows that $a^2 = 9$. So $a = 3, b = 29$.

    Answer: (D)

17. Solution: By the Division Algorithm, $2018 = dq + 8$. So $2010 = dq$. Therefore, d must be a divisor of 2010. Since $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, there are $2^4 = 16$ divisors of 2010. However, d cannot be smaller than 8. Those are $1,2,3,5,6$. Hence the there are $16 - 5 = 11$ numbers that can be d.

    Answer: (B)

18. Solution: Note that $\binom{n}{r}(x^3)^r(x^{-5})^{n-r} = \binom{n}{r}x^{3r}x^{-5n+5r} = \binom{n}{r}x^{8r-5n}$ is the rth term in the expansion. So n and r must satisfy $5n = 8r$ for the expansion to have a constant term. The smallest such pair of numbers is $n = 8$ and $r = 5$. Therefore, the constant term is $\binom{8}{5} = 56$.

    Answer: (C)

19. Solution: Note that $m^2n + mn^2 + m + n = mn(m + n) + m + n = (m + n)(mn + 1) = 77$. So $m + n = 7$ and $mn + 1 = 11$. Solving them for m and n, we have $m = 5$ and $n = 2$.

Answer: (B)

20. Solution: Note that $\frac{N^3+100}{N+4} = \frac{N^3+64+36}{N+4} = N^2 + 4N + 16 + \frac{36}{N+4}$. Therefore, $N + 4$ divides $N^3 + 100$ if and only if $N + 4$ divides 36. Factors of 36 are 1,2,3,4,6,9,12,18,36. Since N is a positive integer, there are five values of $N = 2,5,8,14,32$.

Answer: (C)

21. Solution: $18x = 2 \cdot 3^2 x$ must be a perfect cube number. The smallest such x value is $x = 2^2 \cdot 3 = 12$. Then $y^3 = 216$, so $y = 6$.

Answer: (A)

22. Solution: Let $73p + 1 = n^2$. So $73p = n^2 - 1 = (n - 1)(n + 1)$. Thus $73p$ is a product of two numbers differ by 2. Since 73 and p are prime numbers, $p = 71$.

Answer: (A)

23. Solution: Since $1234ab$ is divisible by 99, it is divisible by 9 and 11. By divisibility rules of 9 and 11, we have $1 + 2 + 3 + 4 + a + b = 10 + a + b$ is divisible by 9, and $1 - 2 + 3 - 4 + a - b = -2 + a - b$ is divisible by 11. A system of equations obtained from them is $a + b = 8, a - b = 2$. Solving the system, we have $a = 5, b = 3$.

Answer: (B)

24. Solution: Note that $n \cdot n! = (n + 1)! - n!$. The sum can be written as $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + 100 \cdot 100! = (2! - 1!) + (3! - 2!) + \cdots + (101! - 100!) = 101! - 1!$.

Since the number $101!$ has 24 0s occurring at the end, $101! - 1$ has 24 9s at the end.

Answer: (C)

25. Solution: Since pqr is divisible by 11, one of p, q, r has to be 11. Let $r = 11$. Then we have $pq = p + q + 11$ by cancelling the common factor 11. Note that $pq - p - q + 1 = (p - 1)(q - 1) =$

12. Since both $p-1$ and $q-1$ are even numbers, $p-1=2, q-1=6$. Thus $p=3, q=7, r=11$.

Answer: (D)

26. Solution: It is clear that $N=2^{2020}$ is divisible by 4. By Euler's Theorem, $2^{\phi(25)}=2^{20}\equiv 1\ (mod\ 25)$ where $\phi(x)$ is the Euler's Totient function representing the number of relatively prime numbers to x less than or equal to x. We have a system of two modular equations. $N\equiv 0\ (mod\ 4)$ and $=(2^{20})^{101}\equiv 1\ (mod\ 25)$. By Chinese Remainder Theorem, $N\ \equiv 76\ (mod\ 100)$.

Answer: (D)

27. Solution: $20!=2^{18}3^{8}5^{3}7^{2}\cdots 19$, so the largest perfect cube dividing 20! Is $N^3=2^{18}3^{6}5^{3}$.

Therefore, $N=2^{6}3^{2}5=2880$.

Answer: (D)

28. Solution: Note that $6^{2018}-4^{1009}=6^{2018}-2^{2018}=2^{2018}(3^{2018}-1)$. $3^{2018}-1$ is divisible by 8 but not divisible by 16. Therefore,$n=2021$ is the largest number such that 2^n divides $6^{2018}-4^{1009}$.

Answer: (B)

29. Solution: Dividing both sides of $ab+bc+ca=abc$ by abc, we have $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$.

There are only three possible triplets of positive integers $(3,3,3), (4,4,2), (2,3,6)$ whose reciprocals sum equals 1. Therefore the largest possible $a+b+c$ value is 11.

Answer: (C)

30. Solution: Note that $n^4-80n^2+100=n^4+20n^2+100-100n^2=(n^2+10)^2-(10n)^2=(n^2+10n+10)(n^2-10n+10)$. If n^4-80n^2+100 is prime, one of $n^2+10n+10$ or $n^2-10n+10$ is equal to 1. But $n^2+10n+10>10$, so we set $n^2-10n+10=1$. Solving the equation, we have $n=1,9$. If $n=1$, $f(1)=1-80+100=21$ is not prime. If $n=9$, $f(9)=9^2+10\cdot 9+10=181$ is prime.

Answer: (B)